An invitation to solve global optimization problems by the use of geometric branch-and-bound methods

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Abstract

Geometric branch-and-bound algorithms are well-known solution techniques in deterministic global optimization. To keep the use of these solution methods as simple as possible, we suggest an easy-to-use JavaScript tool which includes the branch-and-bound algorithm and an interval arithmetic package. Hence, one only has to implement the objective function and the calculation of the lower bounds if one wants to solve a suitable continuous optimization problem. The present work provides an extensive documentation on how to use the code as well as several example problems for instance from the field of location theory which were solved by the use of the JavaScript tool. The suggested code as well as all examples given in this work can be downloaded from our homepage mentioned at the end of the introduction.

Keywords: deterministic global optimization, geometric branch-and-bound, location theory, continuous optimization, JavaScript tool.

1 Introduction

In ***deterministic global optimization*** one wants to find the global minima of some real-valued functions. Many new theoretical and computational contributions to deterministic global optimization have been developed in the last decades and ***geometric branch-and-bound methods*** arose to a commonly used solution technique, in particular if one deals with a few number of continuous variables.

The main task throughout all geometric branch-and-bound algorithms is to calculate lower bounds on the continuous objective function and several methods to do so can be found in the literature: Some general techniques can be found in Horst and Tuy (1996), Horst et al. (2000), Hansen and Jaumard (1995), Horst and Thoai (1999), and Tuy and Horst (1988). Constructing lower bounds by the use of interval analysis was discussed in Ratschek and Rokne (1988), Hansen (1992), and Ratschek and Voller (1991). Moreover, some particular algorithms for facility location problems were suggested in Hansen et al. (1985), Plastria (1992), Drezner and Suzuki (2004), Drezner (2007), Blanquero and Carrizosa (2009), Schöbel and Scholz (2010a), Tuy et al. (1995) and Tuy (1996). For a summary of all these bounding techniques we refer to Scholz (2012a).

Our main intension of the present work is an easy-to-use tool for the solution of global optimization problems by the use of geometric branch-and-bound methods. To this end, following Scholz (2012a), the algorithm was implemented in JavaScript in such a way that one basically only has to specify the objective function and a corresponding bounding operation. JavaScript was chosen since the code can be run on almost every modern browser without any further software, and, furthermore, the code can be edited with any text editor. Hence, the suggested tool can be perfectly used for education purposes.

In the following Section 2 we start with basic notations and a brief summary of the geometric branch-and-bound algorithm. For theoretical results including a convergence theory and for the calculation of lower bounds we refer to Scholz (2012a). Moreover, the section includes a summary of interval analysis, i.e., a technique which provides some simple lower bounds. The main contribution can be found in Section 3 where we give an extensive documentation of the suggested JavaScript tool including an description of the software architecture as well as two first example problems. Furthermore, the section includes a documentation of the interval arithmetic package as well as the problem visualization package which are also included in the JavaScript code. Next, the following Section 4 gives some example problems which can be solved efficiently using the suggested JavaScript geometric branch-and-bound tool. We remark that the JavaScript branch-and-bound tool as well as all examples given in the present work can be downloaded from:

`www.mehr-davon.de/gbb/`

2 Geometric branch-and-bound methods

Following Scholz (2012a), this section summarizes the geometric branch-and-bound prototype algorithm which was implemented in a JavaScript tool as described in Section 3. After some basic notations, we present the general geometric branch-and-bound technique including some further remarks and comments. Finally, we briefly summarize interval arithmetics which might be used to obtain required bounds throughout the algorithm.

2.1 Notations

The following notations are similar to those suggested in Schöbel and Scholz (2010b) and Scholz (2012a).

Notation 1. A compact ***box*** with sides parallel to the axes is denoted by

$$X = [x_1^L, x_1^R] \times \ldots \times [x_n^L, x_n^R] \subset \mathbb{R}^n.$$

The ***diameter*** of a box $X \subset \mathbb{R}^n$ is

$$\delta(X) = \sqrt{\left(x_1^R - x_1^L\right)^2 + \ldots + \left(x_n^R - x_n^L\right)^2}$$

and the ***center*** of a box $X \subset \mathbb{R}^n$ is defined by

$$c(X) = \left(\frac{1}{2}(x_1^L + x_1^R), \ldots, \frac{1}{2}(x_n^L + x_n^R)\right).$$

In the following, we consider the minimization of a continuous function

$$f : X \to \mathbb{R},$$

where we assume a box $X \subset \mathbb{R}^n$ as feasible area, i.e.,

$$X \;=\; [x_1^L, x_1^R] \times \ldots \times [x_n^L, x_n^R] \;\subset\; \mathbb{R}^n.$$

Next, we define bounding operations as follows.

Notation 2. Let $X \subset \mathbb{R}^n$ be a box and consider $f : X \to \mathbb{R}$. A ***bounding operation*** is a procedure to calculate for any subbox $Y \subset X$ a ***lower bound*** $LB(Y) \in \mathbb{R}$ with

$$LB(Y) \;\leq\; f(x) \qquad \text{for all} \quad x \in Y$$

and to specify a point $r(Y) \in Y$.

Several general bounding operations are derived in Schöbel and Scholz (2010b) and Scholz (2012b) and summarized in Scholz (2012a). Moreover, note that the choice of $r(Y)$ is often only important for some theoretical result as discussed in the previously cited references. If it is unclear how to specify $r(Y)$, a general choice is the center of Y, that is,

$$r(Y) \;=\; c(Y).$$

2.2 The algorithm

The general idea of all geometric branch-and-bound algorithms cited in Section 1 is the same: Subboxes of the feasible area are bounded from below making use of a bounding operation as defined before. If the bounds are not sharp enough, some boxes are split into smaller ones according to a given ***splitting rule***. This procedure repeats until the algorithm finds an optimal solution $x^* \in X$ within an absolute accuracy of $\varepsilon > 0$.

To sum up, for the following geometric branch-and-bound algorithm assume an objective function f and a feasible box X. Moreover, we need a bounding operation, a splitting rule, and an absolute accuracy of $\varepsilon > 0$.

```
(1) Let $\mathcal{X}$ be a list of boxes and initialize $\mathcal{X} := \{X\}$.
(2) Apply the bounding operation to $X$ and set $UB := f(r(X))$ and $x^* := r(X)$.
(3) If $\mathcal{X} = \emptyset$, the algorithm stops. Else set
        $\delta_{max} \ := \ \max\{\delta(Y) \, : \, Y \in \mathcal{X}\}.$
(4) Select a box $Y \in \mathcal{X}$ with $\delta(Y) = \delta_{max}$ and split it according to a selected
    splitting rule into $s$ congruent smaller subboxes $Y_1$ to $Y_s$.
(5) Set $\mathcal{X} = (\mathcal{X} \setminus Y) \cup \{Y_1, \ldots, Y_s\}$, i.e., delete $Y$ from $\mathcal{X}$ and add $Y_1, \ldots, Y_s$.
(6) Apply the bounding operation to $Y_1, \ldots, Y_s$ and set
        $UB \ = \ \min\{UB, \, f(r(Y_1)), \, \ldots, \, f(r(Y_s))\}.$
    If $UB = f(r(Y_k))$ for a $k \in \{1, \ldots, s\}$, set $x^* = r(Y_k)$.
(7) For all $Z \in \mathcal{X}$, if $LB(Z) + \varepsilon \geq UB$ set $\mathcal{X} = \mathcal{X} \setminus Z$, i.e., delete $Z$ from $\mathcal{X}$.
(8) Return to Step (3).
```

The selected box Y in Step **(4)** has to be split into s subboxes Y_1 to Y_s according to one of the following ***splitting rules***:

(1) For boxes in small dimensions, say $n \leq 3$, we suggest a split into $s = 2^n$ congruent subboxes.

(2) In higher dimensions, boxes can be bisected perpendicular to the direction of the maximum width component in two subboxes Y_1 and Y_2.

For some theoretical results regarding the termination of the algorithm, we refer to Schöbel and Scholz (2010b) and Scholz (2012a).

2.3 Interval arithmetic

The main task throughout any geometric branch-and-bound algorithm is the calculation of the required lower bounds. Interval arithmetic is a general framework for calculations with intervals, see, e.g., Ratschek and Rokne (1988) or Neumaier (1990), which is also a suitable tool for the calculation of lower bounds as outlined in the textbook Hansen (1992) or in Ratschek and Voller (1991).

Since the following JavaScript tool also includes an interval arithmetic package, this subsection summarizes the basic ideas of interval arithmetic for the calculation of the required bounds as given in Scholz (2012b). Note that we assume compact intervals at all times throughout this paper.

Notation 3. A (compact) interval X is denoted by

$$X \ = \ [a, b] \ \subset \ \mathbb{R}$$

with $a \leq b$. Moreover, the left and right endpoints are denoted by $X^L = a$ and $X^R = b$, respectively.

Next, arithmetic operations between intervals are defined as follows.

Definition 4. Let $X = [a, b]$ and $Y = [c, d]$ be two intervals. Then the ***interval arithmetic*** is given by

$$X \star Y \; := \; \{x \star y \; : \; x \in X, \; y \in Y\},$$

where $\star$ denotes the addition, multiplication, subtraction, division, minimum, or maximum as long as $x \star y$ is defined for all $x, y \in Y$.

By definition, $X \star Y$ again yields an interval which contains $x \star y$ for all $x \in X$ and $y \in Y$ and which can be computed easily. Apart from interval arithmetic also interval operations are defined as follows.

Definition 5. Let $X = [a, b]$ be an interval. Then the ***interval operation*** is given by

$$op(X) \; := \; \{op(x) \; : \; x \in X\} \; = \; \left[\min_{x \in X} op(x), \, \max_{x \in X} op(x)\right],$$

where $op : X \to \mathbb{R}$ denotes a continuous function such that $op(X)$ is an interval.

Definition 6. An ***interval function*** $F(X_1, \ldots, X_n)$ is an interval valued function with n intervals as argument using interval arithmetics and interval operations as defined before.

Example 1. *An interval function* $F(X, Y)$ *with two intervals* X *and* Y *as argument is*

$$F(X, Y) \; = \; \left|\frac{X + Y}{Y^2 + [1, 1]}\right|.$$

For example, we obtain

$$F([0, 2], [-2, 1]) \; = \; \left|\frac{[0, 2] + [-2, 1]}{[-2, 1]^2 + [1, 1]}\right| \; = \; \left|\frac{[-2, 3]}{[1, 5]}\right| \; = \; \left|[-2, 3]\right| \; = \; [0, 3].$$

Definition 7. Let $f(x_1, \ldots, x_n)$ be a fixed representation of a real-valued function with n real numbers as argument using arithmetics and operations such that the corresponding interval arithmetics and interval operations are defined.

Then the ***natural interval extension*** of $f(x_1, \ldots, x_n)$ is given by the interval function $F(X_1, \ldots, X_n)$ where arithmetics and operations are replaced by their corresponding interval arithmetics and interval operations.

Example 2. *The natural interval extension of*

$$f(x, y) \; = \; 4 \cdot x^2 + \frac{\sin(y)}{x^2 + 1}$$

is given by

$$F(X, Y) \; = \; [4, 4] \cdot X^2 + \frac{\sin(Y)}{X^2 + [1, 1]}$$

where X and Y are intervals.

The natural interval extension leads to general lower bounds as required throughout the geometric branch-and-bound algorithm. To this end, we need the following statement which can be found in any standard textbook of interval analysis such as Ratschek and Rokne (1988), Neumaier (1990), or Hansen (1992).

Theorem 1 (Fundamental theorem of interval analysis). *Let $F(X_1, \dots, X_n)$ be the natural interval extension of $f(x_1, \dots, x_n)$. Then*

$$f(Y_1, \dots, Y_n) \ \subseteq \ F(Y_1, \dots, Y_n)$$

for all intervals $Y_k \subseteq X_k$ for $k = 1, \dots, n$, where

$$f(Y_1, \dots, Y_n) \ := \ \{f(x_1, \dots, x_n) \ : \ x_k \in Y_k \text{ for } k = 1, \dots, n\}.$$

Proof. See, for instance, Hansen (1992). □

In other words, if we consider an optimization problem with a real-valued function f, then the natural interval extension F of f yields lower bounds as required throughout the geometric branch-and-bound algorithm, see Subsection 3.5.

On the other hand, we remark the bounds obtained from the natural interval extension might not be sharp enough to solve problems in a reasonable amount of time. For some more sophisticated bounding operations we refer to Scholz (2012b), Schöbel and Scholz (2010b), and Scholz (2012a).

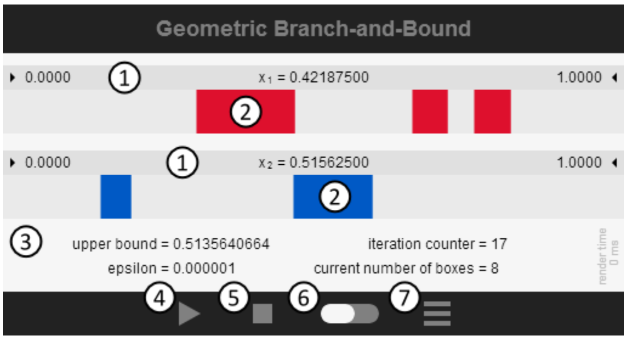
3 Documentation of the JavaScript tool

The suggested geometric branch-and-bound method presented in Section 2 was implemented in JavaScript including an ***interval arithmetic package*** as well as a ***problem visualization package***.

We remark that the biggest advantage of JavaScript is that it can be run on almost every modern internet browser without any further software. Moreover, the code for the problem formulation can be modified with any text editor. Hence, the JavaScript tool can be perfectly used for education purposes which was our main intension throughout this work.

On the other hand, a disadvantage is the run time. Since JavaScript is a scripting language, the code is not compiled and it needs to be interpreted at any time the code is running. Therefore, some much more efficient implementations of the examples given in Section 4 are possible although the JavaScript tool yields adequate results for example for medium-sized facility location problems, see Section 4.

In this section, we first explain the use of the application in a web browser before explaining the software architecture of the proposed JavaScript code where we distinguish between mandatory and optional functions. Next, documentations of the interval arithmetic package



x_1 = 0.42187500

x_2 = 0.51562500

upper bound = 0.5135640664

epsilon = 0.000001

iteration counter = 17

current number of boxes = 8

Figure 1. Control panel of the JavaScript branch-and-bound tool for an example problem with $n = 2$.

and the problem visualization package are given. The section ends with an detailed example problem making use of all the packages explained before.

3.1 The application

The JavaScript tool runs on every modern internet browser and provides a control panel including the elements as well as a visualization of the progress of the branch-and-bound algorithm as given in Figure 1:

(1) For each variable x_1 to x_n, the bounds of the initial box as well as the current best-known solution are displayed.

(2) The colored regions are subboxes which may contain the optimal solution. In other words, if some sub-intervals are not colored, they can not contain any optimal solution (up to the absolute accuracy of ε).

(3) The output text shows the best-known upper bound, the pre-defined absolute accuracy ε, the number of iterations while running the algorithm, and the current number of boxes, i.e., the number of elements in $\mathcal{X}$.

(4) The first button is to start and pause the algorithm.

(5) The second button stops and resets the algorithm. Note that the `init_problem` function described below is called once the button is pressed.

(6) This button is to change between a slow and a fast mode. In the slow mode, the algorithm performs two iterations every second. In the fast mode, the algorithm runs as fast as possible.

(7) The last button deactivates the colored regions. This might be useful if the current number of boxes exceeds some limits which might leed to high rendering times.

3.2 Software architecture

Three files are needed if one wants to solve a minimization problem by the use of the JavaScript branch-and-bound tool, see Figure 2:

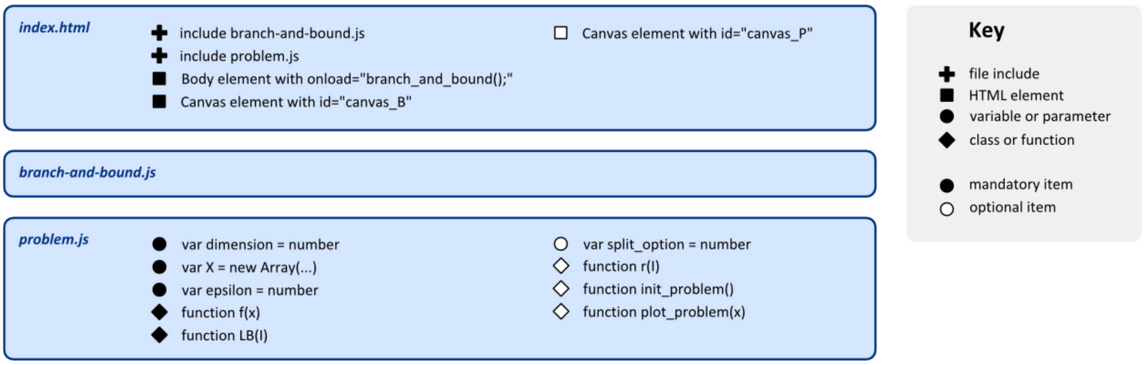


Figure 2. Software architecture of the proposed JavaScript tool.

(1) ***index.html***. This file includes the JavaScript source files and contains the `canvas` element which is needed to illustrate the progress of the branch-and-bound algorithm.

(2) ***branch-and-bound.js***. This file contains the source-code of the JavaScript branch-and-bound tool. The file can be downloaded from

`www.mehr-davon.de/gbb/`

and should not be edited at any time.

(3) ***problem.js***. This file contains the problem definition, i.e., the objective function as well as some mandatory and optional parameters and functions described below.

Mandatory items

There are some mandatory items which should always be used in the `index.html` file, see Figure 2 and Figure 3(a):

(1) Line 4 includes the branch-and-bound tool file `branch-and-bound.js`.

(2) Line 5 includes the problem definition file `problem.js`.

(3) The `onload` statement in line 8 ensures that the branch-and-bound commences at the time the html file is loaded.

(4) The `canvas` element with id `canvas_B` in line 9 is needed to illustrate the progress of the branch-and-bound algorithm and to show the control panel as given in Figure 1.

In addition, the branch-and-bound tool requires the following mandatory variables and functions in the `problem.js` file, see Figure 2 and Figure 3(b):

(1) The variable `dimension` in line 1 defines the dimension of the problem which should be an integer value greater than or equal to one.

(2) The variable `X` in line 2 defines the initial box X of the problem. In the one-dimensional case, `X` is an interval defined by its endpoints. In the higher dimensional case, `X` is an array of intervals.

```
 1  <html>
 2
 3  <head>
 4    <script src="branch-and-bound.js" type="text/javascript"></script>
 5    <script src="problem.js" type="text/javascript"></script>
 6  </head>
 7
 8  <body onload="branch_and_bound();">
 9    <canvas id="canvas_B"></canvas>
10  </body>
11
12  </html>
```

(a) Source-code of the `index.html` file.

```
 1  var dimension = 1;
 2  var X = new Array(0, 8);
 3  var epsilon = 0.000001;
 4
 5  function f (x) {
 6    return Math.sin(x);
 7  }
 8
 9  function LB (I) {
10    return f(0.5*(I[0]+I[1])) - 0.5*(I[1]-I[0]);
11  }
```

(b) Source-code of the `problem.js` file.

Figure 3. One-dimensional example of the JavaScript tool: The objective function $f(x) = \sin(x)$ is minimized over the interval $X = [0, 8]$ with an absolute accuracy of $\varepsilon = 10^{-6}$.

(3) The variable `epsilon` in line 3 defines the absolute accuracy for the branch-and-bound algorithm.

(4) The function `f` in line 5 defines the objective function. Note that the argument `x` is an array starting with index 0 if the dimension is greater than one.

(5) The function `LB` in line 9 specifies the lower bound for any box `Y`. In the one-dimensional case, `Y` is an interval defined by its endpoints. In the higher dimensional case, `Y` is an array of intervals.

Example 3. *Figure 3 presents the source-code of the `index.html` and the `problem.js` files for an example only using the mandatory items as summarized in Figure 2. Here, the one-dimensional objective function*

$$f(x) \quad = \quad \sin(x)$$

is minimized over the box or interval $X = [0, 8]$ with an absolute accuracy of $\varepsilon = 10^{-6}$. The lower bounds are calculated using the Lipschitzian bound operation as presented in Scholz (2012a) and references therein.

The ***one-dimensional*** example problem is to minimize the function

$$f(x) \quad = \quad \sin(x)$$

over the box or interval $X = [0, 8]$.

Optional items

Apart from the mandatory items, the branch-and-bound tool offers some optional features.

In the `index.html` file, there is only one optional item, see Figure 2:

(1) The `canvas` element with id `canvas_P` is needed to illustrate the problem which also requires the function `plot_problem` as described in the following.

In the `problem.js` file, one may use the following options, see Figure 2:

(1) The variable `split_option` defines the splitting rule. If `split_option` has a value of one, boxes are split into 2^n congruent subboxes. Otherwise, boxes are bisected perpendicular to the direction of the maximum width component.

If there is no variable `split_option` defined, the branch-and-bound tool uses the splitting rule as suggested in Subsection 2.2.

(2) The function `r` may be used to define the specific point $r(Y)$ of the bounding operation, see Notation 2. Note that the function should return an array if the dimension is greater than one.

If there is no function `r` defined, the branch-and-bound tool uses of the center of Y.

(3) The function `init_problem` is called to initialize some problem instances before running the branch-and-bound algorithm, see Subsection 3.5.

(4) Finally, the function `plot_problem` is needed to illustrate the problem, see Example 4 and Subsection 3.4.

Example 4. *We want to extend Example 3 in such a way that the objective function is plotted using the problem visualization package described below.*

To this end, the `index.html` *file needs a second* `canvas` *element with id* `canvas_P`*, see line 9 in Figure 4(a). Furthermore, the function* `plot_problem` *illustrates the objective function, see Figure 4(b). For a detailed description of the problem visualization package, we refer to Subsection 3.4.*

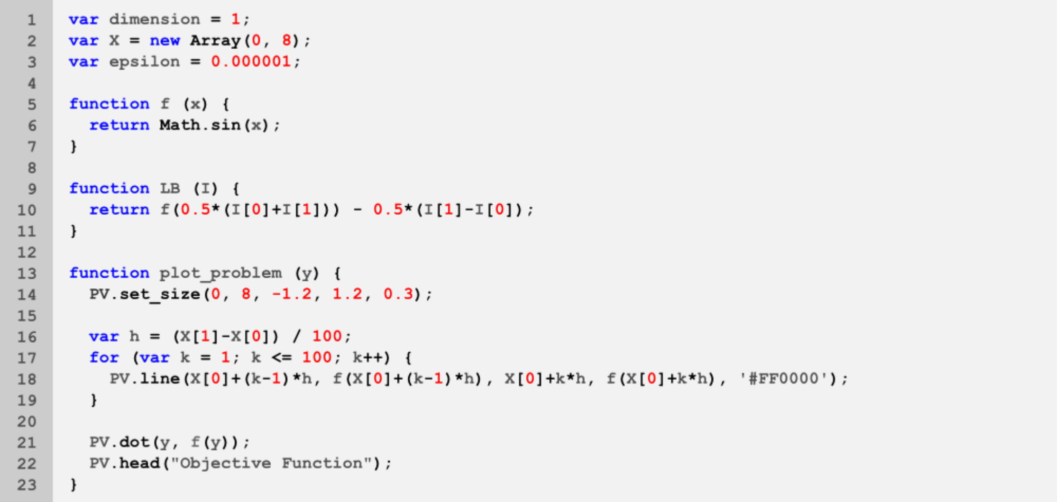
3.3 Interval arithmetic

As indicated in Subsection 2.3, interval arithmetic can be used for the calculation of the required lower bounds. Therefore, the JavaScript tool also includes an interval arithmetic package (`IA`).

This package contains a collection of functions which can be applied to (closed) intervals. Note that the JavaScript code assumes a two-dimensional array as an interval. Table 1 summarizes all functions which are included in the interval arithmetic package. Furthermore, Subsection 3.5 explains exemplarily how to use the package for the calculation of some lower bounds.

```
 1  <html>
 2
 3  <head>
 4    <script src="branch-and-bound.js" type="text/javascript"></script>
 5    <script src="problem.js" type="text/javascript"></script>
 6  </head>
 7
 8  <body onload="branch_and_bound();">
 9    <canvas id="canvas_P"></canvas><br />
10    <canvas id="canvas_B"></canvas>
11  </body>
12
13  </html>
```

(a) Source-code of the `index.html` file.



```
 1  var dimension = 1;
 2  var X = new Array(0, 8);
 3  var epsilon = 0.000001;
 4
 5  function f (x) {
 6    return Math.sin(x);
 7  }
 8
 9  function LB (I) {
10    return f(0.5*(I[0]+I[1])) - 0.5*(I[1]-I[0]);
11  }
12
13  function plot_problem (y) {
14    PV.set_size(0, 8, -1.2, 1.2, 0.3);
15
16    var h = (X[1]-X[0]) / 100;
17    for (var k = 1; k <= 100; k++) {
18      PV.line(X[0]+(k-1)*h, f(X[0]+(k-1)*h), X[0]+k*h, f(X[0]+k*h), '#FF0000');
19    }
20
21    PV.dot(y, f(y));
22    PV.head("Objective Function");
23  }
```

(b) Source-code of the `problem.js` file.

Figure 4. Extension of Example 3 to include a visualization of the objective function $f(x) = \sin(x)$. Apart from the additional line 9, the `index.html` is the similar to the one presented in Figure 3(a). The `problem.js` file was extended by the lines 12–23. Note that the source-code can be downloaded from the homepage mentioned in the introduction.

function	arguments	description and examples
IA.add	interval I, interval J	Returns the sum of two intervals I and J. `IA.add([-1,2],[3,4]) = [2,6]`
IA.sub	interval I, interval J	Returns the difference of two intervals I and J. `IA.sub([-1,2],[3,4]) = [-5,-1]`
IA.mult	interval I, interval J	Returns the product of two intervals I and J. `IA.mult([-1,2],[3,4]) = [-4,8]`

IA.div	interval I, interval J	Returns the quotient of two intervals I and J. If zero is an element of J, the function returns $[0, 0]$.
		`IA.div([-1,4],[2,4]) = [-0.5,2]`
		`IA.div([-1,4],[-1,1]) = [0,0]`

IA.add_scalar	scalar c, interval I	Returns the sum of a scalar c and an interval I. `IA.add_scalar(-1,[2,3]) = [1,2]`
IA.mult_scalar	scalar c, interval I	Returns the product of a scalar c and an interval I. `IA.mult_scalar(-1,[2,3]) = [-3,-2]`

IA.sq	interval I	Returns the square of an interval I.
		`IA.sq([2,3]) = [4,9]`
		`IA.sq([-3,2]) = [0,9]`

IA.sqrt	interval I	Returns the square root of an interval I. Negative values in I are limited from below by zero.
		`IA.sqrt([4,9]) = [2,3]`
		`IA.sqrt([-7,4]) = [0,2]`

IA.pow	interval I, integer p	Returns the power p of an interval I where p is assumed as positive integer. `IA.pow([-1,2],3) = [-1,8]`
IA.min	interval I, interval J	Returns the minimum of two intervals I and J. `IA.min([1,4],[2,3]) = [1,3]`
IA.max	interval I, interval J	Returns the maximum of two intervals I and J. `IA.max([1,4],[2,3]) = [2,4]`
IA.abs	interval I	Returns the absolute value of an interval I. `IA.abs([-1,4]) = [0,4]`
IA.exp	interval I	Returns the exponential of an interval I. `IA.exp([0,2]) = [1,7.389]`
IA.log	interval I	Returns the natural logarithm of an interval I. If zero is an element of I, the function returns $[0, 0]$. `IA.log([1,7.389]) = [0,2]`
IA.sin	interval I	Returns the sine of an interval I. `IA.sin([0,3.141]) = [0,1]`
IA.cos	interval I	Returns the cosine of an interval I. `IA.cos([0,3.142]) = [-1,1]`
IA.atan	interval I	Returns the arctangent of an interval I. `IA.atan([-2,2]) = [-1.107,1.107]`

Table 1. List of all functions included in the interval arithmetic package (`IA`).

function	arguments	description and examples
PV.set_size	scalars x^L, x^R, y^L, y^R, r	Sets the plotting area to $[x^L, x^R] \times [y^L, y^R]$ with a ratio of $r = \lvert y : x \rvert$. This is a **mandatory** function. `PV.set_size(0, 10, 0, 4, 0.4);`
PV.head	string S	Draws the headline / caption of the plotting surface. This is a **mandatory** function. `PV.head("Problem Visualization");`
PV.dot	scalars x, y	Draws a simple black dot at (x, y). `PV.dot(5.0, 2.0);`
PV.disc	scalars x, y, r, color C	Draws a disc with center (x, y), radius r, and color C. `PV.disc(9, 2, 0.5, '#0000FF');`
PV.circle	scalars x, y, r, color C	Draws a circumference / circle line with center (x, y), radius r, and color C. `PV.circle(2, 2, 1.8, '#FF0000');`
PV.line	scalars x_1, y_1, x_2, y_2, color C	Draws a line from (x_1, y_1) to (x_2, y_2) with color C. `PV.line(4, 0.5, 4, 3.5, '#999999');`
PV.rect	scalars x, y, w, h, color C	Draws a rectangle starting at (x, y) with width w, height h, and color C. `PV.rect(6, 0.5, 3.5, 1, '#00A000');`
PV.text	scalars x, y string S	Draws a text box with text S, centered at (x, y). `PV.text(2, 2, "center of circle");`

Table 2. List of all functions included in the problem visualization package (`PV`). Figure 5 illustrates the problem visualization applying all the examples given in this table.

3.4 Problem visualization

Finally, the problem visualization package (`PV`) can be used for the illustration of the problem instance. To this end, a `canvas` element with id `canvas_P` is needed in the `index.html` file as well as the function called `plot_problem` in the `problem.js` file. The argument `y` of the function `plot_problem`, see line 39 in Figure 6(b), is the current best-known solution x^* throughout the algorithm.

Table 2 collects all functions which are included in the problem visualization package, see also Figure 5 and Subsection 3.5. Note that the two functions `set_size` and `head` are mandatory.

3.5 Example problem

As a second example problem, we consider the two-dimensional ***center problem***. Here, we make use of the interval arithmetic package as well as the problem visualization package.

Assuming s given demand points $A_1, \ldots, A_s$ on the plane, we would like to find a circle or disc with smallest radius which contains all these demand points leading to a two-dimensional problem, see Hamacher (1995) and references therein:

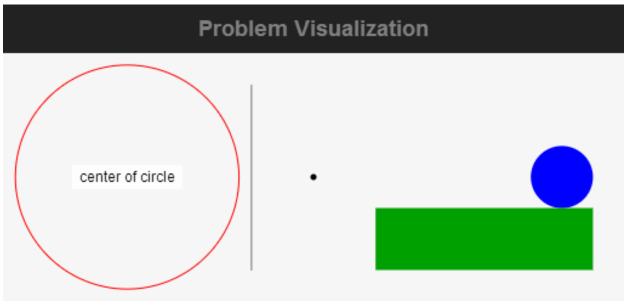


Figure 5. Problem visualization applying all the examples given in Table 2.

> Given some demand points $A_1, \ldots, A_s \in \mathbb{R}^2$, the ***center problem*** is to find the smallest disc with center $X = (x_1, x_2) \in \mathbb{R}^2$ containing these points. Thus, the problem is to minimize the objective function
>
> $$f(x_1, x_2) \;=\; f(X) \;=\; \max_{k=1,\ldots,s} \|A_k - X\|_2$$
>
> where $\| \cdot \|_2$ is the Euclidean norm.

Figure 6 shows the source-code we used to solve the problem. The `index.html` file is the same as outlined in the examples before. The `problem.js` file contains the problem-specific source-code:

(1) The mandatory parameters are defined in lines 1–3.

(2) The `init_problem` function is called once before the algorithm is running. Here, some demand points are randomly distributed in the unit square, see lines 7–11.

(3) Lines 15–23 contain the source-code for the objective function.

(4) Lines 27–35 contain the source-code for the calculation of the lower bounds. Here, we used the natural interval bounding operation, see Scholz (2012a), making use of the interval arithmetic package (`IA`).

(5) Finally, lines 39–47 are to illustrate to problem making use of the problem visualization package (`PV`).

4 Example problems

Facility location problems are to find one or more new locations which, for instance, minimize the sum of some distances to existing demand points. Hence, we obtain a continuous objective function with only a few variables and, therefore, geometric branch-and-bound methods are popular solution techniques.

In this section we summarize some facility location problems as well as geometric problems which were solved by applying the suggested JavaScript branch-and-bound tool, see

`www.mehr-davon.de/gbb/`

```html
<html>

<head>
  <script src="branch-and-bound.js" type="text/javascript"></script>
  <script src="problem.js" type="text/javascript"></script>
</head>

<body onload="branch_and_bound();">
  <canvas id="canvas_P"></canvas><br />
  <canvas id="canvas_B"></canvas>
</body>

</html>
```

(a) Source-code of the `index.html` file.

```javascript
var dimension = 2;
var X = new Array(new Array(0, 1), new Array(0, 1));
var epsilon = 0.000001;

// set random points //

var A = new Array(8);

function init_problem () {
  for (var k = 0; k < A.length; k++) A[k] = new Array(Math.random(), Math.random());
}

// objective function //

function distance (a, x) {
  return Math.sqrt((a[0]-x[0])*(a[0]-x[0]) + (a[1]-x[1])*(a[1]-x[1]));
}

function f (x) {
  var h = distance(A[0], x);
  for (var k = 1; k < A.length; k++) h = Math.max(h, distance(A[k], x));
  return h;
}

// lower bounds by the use of interval arithmetic //

function distance_I (a, I) {
  return IA.sqrt(IA.add(IA.sq(IA.add_scalar(-a[0], I[0])), IA.sq(IA.add_scalar(-a[1], I[1]))));
}

function LB (I) {
  var H = distance_I(A[0], I);
  for (var k = 1; k < A.length; k++) H = IA.max(H, distance_I(A[k], I));
  return H[0];
}

// problem visualization //

function plot_problem (y) {
  PV.set_size(-0.2, 1.2, -0.2, 1.2, 1.0);

  PV.disc(y[0], y[1], f(y), '#E0E0E0');
  for (var k = 0; k < A.length; k++) PV.dot(A[k][0], A[k][1]);
  PV.disc(y[0], y[1], 0.01, '#FF0000');

  PV.head("Problem Visualization");
}
```

(b) Source-code of the `problem.js` file.

Figure 6. Two-dimensional example of the JavaScript branch-and-bound tool: The center problem is solved in $X = [0,1] \times [0,1]$ with an absolute accuracy of $\varepsilon = 10^{-6}$. The source-code can be downloaded from the homepage mentioned in the introduction.

4.1 The Weber problem

The two-dimensional ***Weber problem*** is to find a location for a new facility on the plane which minimizes the weighted sum of Euclidean distances to a given set of demand points. If some weights are positive and others are negative, global optimization techniques are helpful solution algorithms, see for example Schöbel and Scholz (2010a) or Drezner et al. (2001) and references therein.

> Given s demand points $A_1, \ldots, A_s \in \mathbb{R}^2$ with weights $w_1, \ldots, w_s \in \mathbb{R}$, the ***Weber problem*** is to find a new facility $X = (x_1, x_2) \in \mathbb{R}^2$ such that the new facility is close to demand points with positive weights and far away from demand points with negative weights. Thus, the problem is to minimize the objective function
>
> $$f(x_1, x_2) \;=\; f(X) \;=\; \sum_{k=1}^{s} w_k \cdot \|A_k - X\|_2$$
>
> where $\| \cdot \|_2$ is the Euclidean norm.

We applied the JavaScript branch-and-bound tool using the lower bounds as presented in Schöbel and Scholz (2010a).

4.2 The 2-median problem

The ***2-median problem*** or the ***multisource Weber problem*** is to find two new facilities on the Euclidean plane taking some given demand points into account, see Drezner (1984), Chen et al. (1998), or Schöbel and Scholz (2010a). Hence, we are dealing with a four-dimensional problem which can be formulated as follows.

> Given s demand points $A_1, \ldots, A_s \in \mathbb{R}^2$, the ***2-median problem*** is to find two new facilities
>
> $$X_1 \;=\; (x_1, x_2) \;\in\; \mathbb{R}^2 \qquad \text{and} \qquad X_2 \;=\; (x_3, x_4) \;\in\; \mathbb{R}^2$$
>
> such that every demand point is served by its nearest new facility. Hence, we want to minimize the objective function
>
> $$f(x_1, x_2, x_3, x_4) \;=\; f(X_1, X_2) \;=\; \sum_{k=1}^{s} \min\{\|A_k - X_1\|_2,\, \|A_k - X_2\|_2\}$$
>
> where $\| \cdot \|_2$ is the Euclidean norm.

We applied the JavaScript branch-and-bound tool using the lower bounds as presented in Schöbel and Scholz (2010a).

4.3 The 3-median problem

The ***3-median problem*** is an extension of the 2-median problem to three new facilities, see Drezner (1984), Chen et al. (1998), or Schöbel and Scholz (2010a). Thus, we are dealing with the following six-dimensional problem.

> Given s demand points $A_1, \ldots, A_s \in \mathbb{R}^2$, the ***3-median problem*** is to find three new facilities
>
> $$X_1 = (x_1, x_2), \qquad X_2 = (x_3, x_4), \qquad \text{and} \qquad X_3 = (x_5, x_6)$$
>
> such that every demand point is served by its nearest new facility. Hence, we want to minimize the objective function
>
> $$f(x_1, x_2, x_3, x_4, x_5, x_6) = f(X_1, X_2, X_3) = \sum_{k=1}^{s} \left(\min_{j=1,2,3} \|A_k - X_j\|_2 \right)$$
>
> where $\| \cdot \|_2$ is the Euclidean norm.

We applied the JavaScript branch-and-bound tool again using the lower bounds as presented in Schöbel and Scholz (2010a).

4.4 The median circle problem

The ***median circle problem*** is to locate a circle so as to minimize the sum of distances between the circumference and some given demand points on the plane, see Brimberg et al. (2009) or Schöbel and Scholz (2010a) for more details. Since every circle can be defined by its center and radius, we have a three-dimensional problem.

> Given s demand points $A_1, \ldots, A_s \in \mathbb{R}^2$, the ***median circle problem*** is to find the center $X = (x_1, x_2)$ and radius $x_3 \geq 0$ of a circle such that the sum of distances between the circumference and the demand points is minimized. Hence, we find the objective function
>
> $$f(x_1, x_2, x_3) = f(X, x_3) = \sum_{k=1}^{s} \Big| \|A_k - X\|_2 - x_3 \Big|$$
>
> where $\| \cdot \|_2$ is the Euclidean norm.

We applied the JavaScript branch-and-bound tool using the lower bounds as presented in Schöbel and Scholz (2010a).

4.5 The circle detection problem

Global optimization techniques can also be used in image processing to detect imperfect pictured shapes such as lines, circles, and ellipses. We here present the ***circle detection***

problem although in the same manner it is also possible to detect other shapes such as lines or ellipses. Note that one first has to detect edges in a given image before the objective function can be formulated as follows, see Breuel (2003a), Breuel (2003b), or Scholz (2012a).

Assume a set of points $A_1, \dots, A_s \in \mathbb{R}^2$, for example derived from an edge detection algorithm of an image. Than the ***circle detection problem*** is to find the center $X = (x_1, x_2)$ and radius $x_3 \geq 0$ of a circle such that it fits to the given points. This problem can be modeled by a minimization problem with objective function

$$f(x_1, x_2, x_3) \;=\; -\sum_{k=1}^{s} \exp\left(-\frac{1}{\delta} \cdot (\|A_k - X\|_2 - x_3)^2\right) + C \cdot x_3$$

where $\| \cdot \|_2$ is the Euclidean norm and $\delta > 0$, $C \geq 0$ are some regularization parameters.

We applied the JavaScript branch-and-bound tool using the lower bounds as presented in Scholz (2012a).

4.6 The rigid body transformation problem

The two-dimensional ***rigid body transformation problem*** is to align two sets of points in the Euclidean space for which correspondence is known, see Eggert et al. (1997) and references therein. The transformation is defined by a rotation matrix and a translation vector.

Assume two sets of points

$$A_1, \dots, A_s \in \mathbb{R}^2 \qquad \text{and} \qquad B_1, \dots, B_s \in \mathbb{R}^2$$

in such a way that A_k is assigned to B_k for $k = 1, \dots, s$. Than the two-dimensional least squares ***rigid body transformation problem*** is to minimize the objective function

$$f(x_1, x_2, x_3) \;=\; \sum_{k=1}^{s} \left\| \begin{pmatrix} \cos(x_1) & \sin(x_1) \\ -\sin(x_1) & \cos(x_1) \end{pmatrix} \cdot A_k + \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} - B_k \right\|_2^2$$

where $\| \cdot \|_2^2$ is the squared Euclidean norm.

We applied the JavaScript branch-and-bound tool using the general bounding operation of second order, see Scholz (2012a). Note that the same procedure can also be applied to higher dimensional problems.

5 Conclusions

Geometric branch-and-bound methods are well-known solution techniques in deterministic global optimization. Our aim was to provide some software such that global optimization problems can be solved as simple as possible by the use of a geometric branch-and-bound algorithm. To this end, we suggested a JavaScript tool which includes the branch-and-bound algorithm as well as an interval arithmetic package and a problem visualization package. Hence, one basically only has to implement the objective function and a rule for the calculation of the lower bounds if one wants to solve a global optimization problem. For the calculation of the required bounds, one might use the interval arithmetic package, see for example Subsection 3.5.

Although the JavaScript code can be used very easily and no further software is needed, the disadvantage is the run time since JavaScript code needs to be interpreted at any time the code is running. On the other hand, in Section 4 several examples from the field of location theory are given which can be solved efficiently for small and medium-sized problem instances. Furthermore, the code of the JavaScript tool as well as all example problems can be downloaded from the homepage mentioned in the introduction.

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